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On a zero-range interaction of a quantum particle with the vacuum

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Abstract. Self-adjoint extensions of the operator-Δ with the domain $C_0^\infty(\mathbb{R}^3)$ in the space $C^k \oplus L_2(\mathbb{R}^3)$ are described. Such operators are interpreted as Hamiltonians of a point interaction of a quantum particle with the vacuum. Bound states and scattering objects of these Hamiltonians are investigated.

1. Introduction

A zero-range interaction (e.g. see [1-3]) is one of the most popular solvable models of quantum mechanics. It is well adapted for description of a short-range interaction of two quantum particles or, equivalently, of a quantum particle with an external field at low (at least, not too high) energies. Hamiltonians H of a zero-range interaction are constructed in the Hilbert space $L_2(\mathbb{R}^3)$ by the following explicit procedure. Suppose that $u \in L_2(\mathbb{R}^3)$ is a smooth function outside of any neighbourhood of $x=0$ and has the asymptotics

$$u(x) \sim u^{(0)}|x|^{-1} + u^{(1)} \qquad u^{(j)} \in \mathbb{C} \tag{1.1}$$

as $x \to 0$. Then $H = H(\alpha)$ is defined by the formula $(Hu)(x) = -(\Delta u)(x), x \neq 0$, on functions u satisfying (1.1) with coefficients $u^{(j)}$ related by $u^{(1)} = \alpha u^{(0)}$, $\alpha \in \mathbb{R}$. For the boundary condition $u^{(0)} = 0$ the singularity of (1.1) at $x=0$ disappears and we obtain the Hamiltonian $H_0 = -\Delta$ of a free particle. For any $\alpha \in \mathbb{R}$ a zero-range 'potential' is always negative. If $\alpha \geqslant 0$ the 'depth' of a zero-range well is not sufficient to bind a particle and such a well contains exactly one bound state with energy $-\alpha^2$ if $\alpha < 0$. Note that the Hamiltonians $H(\alpha)$ and H_0 are self-adjoint extensions of the symmetric operator $\mathring{H} = -\Delta$ defined on functions $u(x)$ vanishing at $x=0$ (so that $u^{(0)} = u^{(1)} = 0$ in (1.1)). All self-adjoint extensions of $\mathring{H}$ are exhausted by the family $H(\alpha)$ and H_0.

In the present paper we give a similar construction in the case where an additional one-dimensional (or, more generally, finite-dimensional) space is added to $L_2(\mathbb{R}^3)$. Thus, we consider Hamiltonians generated in the space $\mathbb{C} \oplus L_2(\mathbb{R}^3)$ by the differential operator $-\Delta$ and some boundary condition at $x=0$. Actually, we distinguish two families of operators $H_0 = H_0(A_0, a_0)$ and $H = H(A, a, \alpha)$ parametrized by real numbers A_0, A, α and complex numbers a_0, a. Hamiltonians H_0 and H are defined

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on elements $\boldsymbol{u}=\{\xi, u\}$, where $\xi\in\mathbb{C}$ and a function $u(x)$ has the asymptotics (1.1) as $x\to 0$. Coefficients ξ, $u^{(0)}$, $u^{(1)}$ are related by the formulae

$$u^{(0)}=-\bar{a}_0\xi \text{ (for } H_0) \qquad \text{and} \qquad u^{(1)}=\bar{a}\xi+\alpha u^{(0)} \qquad \text{(for } H)$$

where the overbar means complex conjugation. It turns out that the operators introduced by the equalities

$$H_0\boldsymbol{u}=\{A_0\xi+a_0u^{(1)}, -\Delta u\} \qquad H\boldsymbol{u}=\{A\xi+au^{(0)}, -\Delta u\}$$

are self-adjoint in the space $\mathbb{C}\oplus L_2(\mathbb{R}^3)$. In the case $a=0$ or $a_0=0$ the Hamiltonians H and H_0 are decomposed into orthogonal sums of operators in $\mathbb{C}$ and $L_2(\mathbb{R}^3)$. Actually, $H(A, 0, \alpha)=A\oplus H(\alpha)$, where $H(\alpha)$ is the operator defined in the previous paragraph. Thus, setting $a=0$, we recover the Hamiltonians of zero-range interaction in the space $L_2(\mathbb{R}^3)$. Similarly, in the case $a_0=0$ we have that $H_0(A_0, 0)=A_0\oplus H_0$, where H_0 is the 'free' Hamiltonian.

Operators $H_0=H_0(A_0, a_0)$ and $H=H(A, a, \alpha)$ can serve as model Hamiltonians for a description of interaction of a quantum particle with an external *quantized* field at low or moderate energies. Since the number of particles in such a process is not conserved, the problem should be formulated in the Fock space. In the model introduced above, the operators act in the space $\mathbb{C}\oplus L_2(\mathbb{R}^3)$ with the vacuum and one-particle sectors only. Thus, possible annihilation of a particle is taken into account but creation of two or more particles is neglected. Such an approximation seems to be reasonable for low energies and, anyway, this is a price that we have to pay for solvability of the model suggested. Constants A_0, A correspond to interaction of the vacuum with itself, a_0, a describe the point interaction of a particle with the vacuum and α is the depth of zero-range potential well. So by means of the boundary condition we have introduced non-trivial interaction of a particle with the vacuum.

Actually, we consider the somewhat more general situation where operators H_0 and H act in the space $\mathbb{C}^k\oplus L_2(\mathbb{R}^3)$ with arbitrary finite k. In this case A_0, A are self-adjoint matrices and a_0, a are vectors in $\mathbb{C}^k$. In our interpretation the case $k>1$ corresponds to the degeneracy of the vacuum. Operators H_0 and H are self-adjoint extensions in the space $\mathbb{C}^k\oplus L_2(\mathbb{R}^3)$ of the operator $\mathring{H}$ defined by the formula $\mathring{H}\boldsymbol{u}=\{0, -\Delta u\}$ on elements $\boldsymbol{u}$ such that $\xi=u^{(0)}=u^{(1)}=0$. We emphasize that the operator $\mathring{H}$ is not densely defined but the domains of the Hamiltonians H_0 and H are dense in the space $\mathbb{C}^k\oplus L_2(\mathbb{R}^3)$.

This paper was inspired by the works of Yu M Shirokov [4, 5], where highly singular potentials were considered in the framework of the theory of distributions. A correct mathematical interpretation to [4, 5] was given by B S Pavlov and his co-authors (e.g. see [6–8]). Our intention here is to find explicit formulae for all Hamiltonians of zero-range interaction. In particular, we show that the problem of description of self-adjoint extensions of the operator $\mathring{H}$ in the space $\mathbb{C}^k\oplus L_2(\mathbb{R}^3)$ admits quite an elementary and complete solution. In fact, all such self-adjoint extensions are exhausted by the operators $H_0=H_0(A_0, a_0)$ and $H(A, a, \alpha)$ introduced above. Therefore, zero-range Hamiltonians are described, up to some finite-dimensional term, by the same differential operator $-\Delta$ and the boundary condition at $x=0$, which 'couples' the vacuum and the one-particle sectors. Furthermore, we give explicit formulae for quadratic forms of the Hamiltonians H_0 and H. This is very useful for a qualitative analysis of their discrete spectra.

From a technical point of view we rely on the spherical symmetricity of the problem. Let us first explain this point using the example of self-adjoint extensions of $\mathring{H}$ in the space $L_2(\mathbb{R}^3)$. Separating the radial variable $r=|x|$, we reduce the problem to the consideration of the family of the operators $-d^2/dr^2+l(l+1)r^{-2}$, where $l=0,1,2,\dots$ is the orbital quantum number, with the domains $C_0^\infty(\mathbb{R}_+)$ in the space $L_2(\mathbb{R}_+)$. The operators for $l\geq 1$ are essentially self-adjoint so that it suffices to describe self-adjoint extensions of the operator $-d^2/dr^2$ only. All these extensions are parametrized by the boundary condition $u'(0)=\alpha u(0)$, $\alpha\in\mathbb{R}$ or $u(0)=0$. Returning to three-dimensional notation, we obtain the operators $H(\alpha)$ or H_0 introduced in the first paragraph. Similarly, the construction of self-adjoint extensions of $\mathring{H}$ in the space $\mathbb{C}^k\oplus L_2(\mathbb{R}^3)$ reduces to the same problem for the operator $\mathring{H}=-d^2/dr^2$ with the domain $C_0^\infty(\mathbb{R}_+)$ in the space $\mathbb{C}^k\oplus L_2(\mathbb{R}_+)$.

The main motivation for writing this paper was the following curious observation. Suppose that $k=1$ and (as is physically reasonable) that a vacuum–vacuum interaction is zero. The operator $H(0,0,\alpha)\geq 0$ for $\alpha\geq 0$ but, for arbitrary $a\neq 0$, the operator $H(0,a,\alpha)$ has a negative eigenvalue. Thus, even if a zero-range potential well is so shallow that it does not bind a particle, a bound state arises as an arbitrary weak interaction when the vacuum is switched on. On the other hand, the operator $H_0(0,\alpha_0)\geq 0$ for all $a_0\in\mathbb{C}$. This means that, in the absence of a potential well, an interaction with the vacuum never binds a particle.

This paper is organized as follows. In section 2 we describe all self-adjoint extensions H and H_0 of the operator $\mathring{H}$ in the space $\mathbb{C}^k\oplus L_2(\mathbb{R}_+)$. The main result is formulated as theorem 1. Its proof is given in section 3. In section 4 we give expressions for quadratic forms of the constructed Hamiltonians and analyse their discrete spectra. In section 5, resolvents and scattering matrices are calculated and quantative information about eigenvalues is obtained. Finally, in section 6, the results of previous sections are reformulated in terms of the representation in the space $\mathbb{C}^k\oplus L_2(\mathbb{R}^3)$.

2. The zero-range Hamiltonians

Let $\mathscr{H}=\mathbb{C}^k\oplus L_2(\mathbb{R}_+)$ be the 'truncated' Fock space with only the vacuum (possibly degenerate) and the one-particle sectors. Thus, elements of $\mathscr{H}$ are couples $\boldsymbol{u}=\{\xi,u\}$ where $\xi\in\mathbb{C}^k$, $u\in L_2(\mathbb{R}_+)$. The scalar products in $\mathscr{H}$ and $\mathbb{C}^k$ are denoted by $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$, respectively. Clearly, $L_2=L_2(\mathbb{R}_+)$ can be considered as the subspace of $\mathscr{H}$ if elements $u\in L_2$ and $\boldsymbol{u}=\{0,u\}\in\mathscr{H}$ are identified. Hamiltonians corresponding to a zero-range interaction of a particle with the vacuum are introduced as self-adjoint extensions of the symmetric operator $\mathring{H}=-d^2/dr^2$ defined on the domain $\mathscr{D}(\mathring{H})=C_0^\infty$ (the set $C_0^\infty=C_0^\infty(\mathbb{R}_+)$ consists of functions vanishing in some neighbourhoods of zero and infinity). The domain $\mathscr{D}(\mathring{H})$ is not, of course, dense in $\mathscr{H}$. We shall construct all self-adjoint (densely defined) operators $\boldsymbol{H}$ in the space $\mathscr{H}$ such that $\mathscr{D}(\mathring{H})\subset\mathscr{D}(\boldsymbol{H})$ and $\boldsymbol{H}\boldsymbol{u}=\mathring{H}\boldsymbol{u}$ if $\boldsymbol{u}\in\mathscr{D}(\mathring{H})$.

Denote by $W_2^2=W_2^2(\mathbb{R}_+)$ the Sobolev space of functions which belong to L_2 with its two derivatives. For functions $u\in W_2^2$ the boundary values $u(0)$ and $u'(0)$ are correctly defined. Let $\mathring{W}_2^2$ consist of those $u\in W_2^2$ for which $u(0)=u'(0)=0$. Clearly, the closure of $\mathring{H}$ is defined on $\mathring{W}_2^2$. We are now able to formulate our main result.

Theorem 1. Suppose that A is some self-adjoint operator in $\mathbb{C}^k$, vector $a\in\mathbb{C}^k$ and $\alpha\in\mathbb{R}$. Let $\mathscr{D}(\boldsymbol{H})\subset\mathbb{C}^k\oplus W_2^2$ consist of elements $\boldsymbol{u}=\{\xi,u\}$ such that

$$u'(0)=\langle\xi,a\rangle+\alpha u(0). \tag{2.1}$$

Then the operator $H = H(A, a, \alpha)$ defined by the formula

$$Hu = \{A\xi + u(0)a, -u''\} \tag{2.2}$$

is self-adjoint and $\mathring{H} \subset H$. Suppose that A_0 is some self-adjoint operator in $\mathbb{C}^k$ and vector $a_0 \in \mathbb{C}^k$. Let $\mathcal{D}(H_0) \subset \mathbb{C}^k \oplus W_2^2$ consist of elements u such that

$$u(0) = -\langle \xi, a_0 \rangle. \tag{2.3}$$

Then the operator $H_0 = H_0(A_0, a_0)$ defined by the formula

$$H_0 u = \{A_0\xi + u'(0)a_0, -u''\} \tag{2.4}$$

is self-adjoint and $\mathring{H} \subset H_0$. On the other hand, any self-adjoint extension of the operator $\mathring{H}$ in the space $\mathcal{H}$ has one of these two forms.

We emphasize that the operators H and H_0 are determined by non-diagonal matrices. For example,

$$H = \begin{pmatrix} A & \delta(\cdot)a \\ 0 & -\mathrm{d}^2/\mathrm{d}r^2 \end{pmatrix} \qquad \delta(u) = u(0)$$

where u is written as a column. Nevertheless, such an operator is self-adjoint due to boundary condition (2.1) which 'couples' $\mathbb{C}^k$ and L^2 components of $u \in \mathcal{D}(H)$.

Operators H and H_0 are called zero-range Hamiltonians here. Vectors $a \in \mathbb{C}^k$ or $a_0 \in \mathbb{C}^k$ describe a point interaction of a particle with the (degenerate) vacuum. If $a = 0$ or $a_0 = 0$, then the Hamiltonians H or H_0 are decomposed into orthogonal sums of two operators acting in $\mathbb{C}^k$ and $L_2(\mathbb{R}_+)$. Operators in $\mathbb{C}^k$ are determined, of course, by $k \times k$ Hermitian matrices A or A_0. The one-particle operator is $-\mathrm{d}^2/\mathrm{d}r^2$ with the boundary condition $u'(0) = \alpha u(0)$ (for H) or $u(0) = 0$ (for H_0). The first case corresponds to an interaction of a particle with a zero-range potential well. For the Dirichlet boundary condition $u(0) = 0$ the depth of the zero-range potential well is zero so that such an operator describes a free particle.

Note that the matrices A and A_0 can be chosen to be diagonal. Let, for example, $A = T^*\tilde{A}T$ with a unitary $T: \mathbb{C}^k \to \mathbb{C}^k$ and a diagonal matrix $\tilde{A} = \mathrm{diag}\{\lambda^{(j)}\}$. Set $\tilde{a} = Ta$. Then Hamiltonians H parametrized by A, a, α and $\tilde{A}, \tilde{a}, \alpha$ are unitarily equivalent:

$$H(A, a, \alpha) = \Phi H(\tilde{A}, \tilde{a}, \alpha)\Phi^*$$

where Φ,

$$\Phi\{\xi, u\} = \{T^*\xi, u\} \tag{2.5}$$

is a gauge transformation. Furthermore, we can take into account that the relation $A = T^*\tilde{A}T$ holds if T is replaced by ΛT, where $\Lambda = \mathrm{diag}\{\exp(\mathrm{i}\varphi^{(j)})\}$ and real numbers $\varphi^{(j)}$ are arbitrary. Thus, we can always obtain a vector $\tilde{a} = Ta$ with non-negative components. Therefore, up to a canonical unitary equivalence, Hamiltonians H are parametrized by k real numbers $\lambda^{(j)}$, k non-negative numbers $\tilde{a}^{(j)}$ and a real number α corresponding to a point interaction in $L_2(\mathbb{R}_+)$. In particular, in the case $k = 1$ the zero-range interaction of a particle with the vacuum is described by a complex constant a which, due to the gauge transformation, can be chosen non-negative. Similarly, if $A_0 = T^*\tilde{A}_0T$ and $\tilde{a}_0 = Ta_0$, then

$$H_0(A_0, a_0) = \Phi H_0(\tilde{A}_0, \tilde{a}_0)\Phi^*$$

where Φ is again defined by (2.5). The matrix $\tilde{A}_0$ can, of course, be chosen to be diagonal and vector $\tilde{a}_0$ to have non-negative components.

We emphasize that if $a\,(a_0)$ is an eigenvector of $A\,(A_0)$ then the Hamiltonian $H\,(H_0)$ is decomposed into an orthogonal sum of an operator of the same type in the space $\mathbb{C}\oplus L_2(\mathbb{R}_+)$ and of a self-adjoint operator in $\mathbb{C}^{k-1}$. In the general case a reduction to the case $k=1$ is not possible.

In the space $\mathscr{H}=L_2(\mathbb{R}_+)$ the Hamiltonian $H(\alpha)$ generated by the differential operator $-\mathrm{d}^2/\mathrm{d}r^2$ and the boundary condition $u'(0)=\alpha u(0)$ converges as $|\alpha|\to\infty$ to the free operator H_0 for which $u(0)=0$. More precisely, for any complex z the resolvent of $H(\alpha)$ converges in the topology of the norm to that of H_0. It is allowed here that either $\alpha\to\infty$, i.e. the depth of the zero-range potential well tends to zero, or $\alpha\to-\infty$, i.e. the depth tends to infinity. A similar assertion holds in the space $\mathscr{H}=\mathbb{C}^k\oplus L_2(\mathbb{R}_+)$. In fact, Hamiltonian (2.3), (2.4) can be obtained as some limit of Hamiltonians (2.1), (2.2). This procedure requires, in particular, a vacuum renormalization. The proof of the following assertion will be given in section 5.

Proposition 1. Let

$$H_0=H_0(A_0,a_0),\ A_\alpha=A_0+\alpha\langle\cdot,a_0\rangle a_0,\ a_\alpha=\alpha a_0$$

and

$$H_\alpha=H(A_\alpha,a_\alpha,\alpha).$$

Then

$$\lim_{|\alpha|\to\infty}\|(H_\alpha-z)^{-1}-(H_0-z)^{-1}\|=0 \qquad \operatorname{Im} z\neq 0. \tag{2.6}$$

3. Proof of theorem 1

Here we describe all symmetric (densely defined) and, in particular, self-adjoint extensions $\boldsymbol{H}$ of the operator $H_0=-\mathrm{d}^2/\mathrm{d}r^2$ with the domain $\mathscr{D}(\mathring{H})=C_0^\infty(\mathbb{R}_+)$ in the space $\mathscr{H}=\mathbb{C}^k\oplus L_2(\mathbb{R}_+)$. In this section we denote the $\mathbb{C}^k$ component of $\boldsymbol{u}=\{\xi,u\}$ by u_0, i.e. we set $u_0=\xi$. We start with the following simple observation.

Lemma 1. Suppose that for some $\boldsymbol{v},\boldsymbol{w}\in\mathscr{H}$ and arbitrary $\boldsymbol{u}\in\mathscr{D}(\mathring{H})$

$$(\mathring{\boldsymbol{H}}\boldsymbol{u},\boldsymbol{v})=(\boldsymbol{u},\boldsymbol{w}). \tag{3.1}$$

Then $v\in W_2^2$ and $w=-v''$.

Proof. Since $u_0=0$ and $(\boldsymbol{H}\boldsymbol{u})_0=0$, (3.1) implies that

$$-\int_0^\infty u''(r)\overline{v(r)}\,\mathrm{d}r=\int_0^\infty u(r)\overline{w(r)}\,\mathrm{d}r$$

for arbitrary $u\in C_0^\infty$. It follows that u has two (distributional) derivatives and $w=-v''$. Moreover, $v\in W_2^2$ because $w\in L_2$. □

Let us now find necessary conditions on symmetric extensions $\boldsymbol{H}$ of $\mathring{H}$.

Lemma 2. Let $\mathring{H}\subset\boldsymbol{H}\subset\boldsymbol{H}^*$. Then

$$\mathscr{D}(\boldsymbol{H})\subset\mathbb{C}^k\oplus W_2^2(\mathbb{R}_+)=:\mathscr{D}_* \tag{3.2}$$

and there exist a linear operator $L: \mathbb{C}^k \to \mathbb{C}^k$ and vectors $l, \tilde{l} \in \mathbb{C}^k$ such that

$$\boldsymbol{H}\boldsymbol{u} = \{Lu_0 + u(0)l + u'(0)\tilde{l}, -u''\} \tag{3.3}$$

for any $\boldsymbol{u} = \{u_0, u\} \in \mathscr{D}(\boldsymbol{H})$.

Proof. Inclusion (3.2) is an immediate consequence of lemma 1. Furthermore, according to lemma 1 the L_2 component of $\boldsymbol{Hu}$ equals $-u''$. To find $(\boldsymbol{Hu})_0$ we denote by φ and θ some smooth functions on $\mathbb{R}_+$ with compact supports such that $\varphi(0) = \theta'(0) = 0$, $\varphi'(0) = \theta(0) = 1$. An arbitrary function $u \in W_2^2$ can be decomposed uniquely into a sum

$$u = \alpha\varphi + \beta\theta + \mathring{u} \qquad \mathring{u} \in \mathring{W}_2^2 \tag{3.4}$$

where $\alpha = u'(0)$, $\beta = u(0)$. Therefore, every $\boldsymbol{u} \in \mathscr{D}(\boldsymbol{H})$ has a form $\boldsymbol{u} = \{u_0, u'(0)\varphi + u(0)\theta\} + \mathring{\boldsymbol{u}}$ with $\mathring{\boldsymbol{u}} \in \mathscr{D}(\mathring{H})$. Since $(\boldsymbol{H}\mathring{\boldsymbol{u}})_0 = 0$, the $\mathbb{C}^k$ component of $\boldsymbol{Hu}$ depends on u_0, $u(0)$ and $u'(0)$ only. This ensures representation (3.3). □

According to (3.4) the factor space $X_* = \mathscr{D}_*/\mathscr{D}(\mathring{H})$ consists of vectors $\{u_0, u(0), u'(0)\}$ and has dimension $k+2$. By (3.2), $\mathscr{D}(\boldsymbol{H}) = \mathscr{D}(\mathring{H}) \dot{+} X$, where X is some linear subspace of X_*. Let $J: \{u_0, u(0), u'(0)\} \mapsto u_0$ be the natural projection of M_* onto $\mathbb{C}^k$. It is easy to see that $\overline{\mathscr{D}(\boldsymbol{H})} = H$ if and only if $JX = \mathbb{C}^k$. Indeed, suppose that

$$\langle f_0, u_0 \rangle + \int_0^\infty f(x)\overline{u(x)}\,\mathrm{d}x = 0 \tag{3.5}$$

for some $f \in \mathscr{H}$ and any $\boldsymbol{u} \in \mathscr{D}(\boldsymbol{H})$. In particular, choosing $\boldsymbol{u} \in \mathscr{D}(\mathring{H})$ we find that $f = 0$ so that (3.5) is equivalent to $\langle f_0, u_0 \rangle = 0$. Clearly, $f_0 = 0$ if and only if $u_0 \in \mathbb{C}^k$ is arbitrary. Thus, we assume below that $JX = \mathbb{C}^k$. It follows that $\dim X = k$.

Suppose now that an operator $\boldsymbol{H}$ is defined by (3.3) on a domain $\mathscr{D}(\boldsymbol{H}) = \mathscr{D}(\mathring{H}) \dot{+} X$. Let us construct its adjoint $\boldsymbol{H}^*$. Assume that for some $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}$ and arbitrary $\boldsymbol{u} \in \mathscr{D}(\boldsymbol{H})$

$$(\boldsymbol{Hu}, \boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{w}) \qquad \boldsymbol{w} = \boldsymbol{H}^*\boldsymbol{v}. \tag{3.6}$$

By lemma 1, $w = -v''$ and integrating by parts, we obtain the equality

$$\langle (\boldsymbol{Hu})_0, v_0 \rangle - \langle u_0, w_0 \rangle - u(0)\overline{v'(0)} + u'(0)\overline{v(0)} = 0 \tag{3.7}$$

which is equivalent to (3.6).

Below, we study the cases $\dim X = k, k+1, k+2$ separately. If $\dim X = k$, then the values of u_0, $u(0)$ and $u'(0)$ for $\boldsymbol{u} \in \mathscr{D}(\boldsymbol{H})$ are connected by the two relations

$$u(0) = \langle u_0, b \rangle \qquad u'(0) = \langle u_0, \tilde{b} \rangle \qquad b, \tilde{b} \in \mathbb{C}^k. \tag{3.8}$$

Thus, (3.3) is reduced to $\boldsymbol{Hu} = \{Bu_0, -u''\}$, where $B(B = L + \langle \cdot, b \rangle l + \langle \cdot, \tilde{b} \rangle \tilde{l})$ is some linear operator in $\mathbb{C}^k$. Since $u_0 \in \mathbb{C}^k$ is arbitrary, (3.7) determines

$$w_0 = B^* v_0 - v'(0)b + v(0)\tilde{b}. \tag{3.9}$$

The inclusion $\boldsymbol{H} \subset \boldsymbol{H}^*$ is equivalent to the relation $w_0 = Bv_0$ for $\boldsymbol{v} \in \mathscr{D}(\boldsymbol{H})$. By (3.8), (3.9) this relation holds if and only if the operator $B + \langle \cdot, \tilde{b} \rangle b$ is self-adjoint. Under this assumption the operator $\boldsymbol{H}$ is symmetric. However, if w_0 is defined by (3.9), then (3.7) and therefore (3.6) are fulfilled for arbitrary $v \in \mathscr{D}_*$. Thus, $\mathscr{D}(\boldsymbol{H}^*) = \mathscr{D}_*$ so that $\boldsymbol{H}$ is not self-adjoint.

Further, let $\dim X = k+1$. Then $u_0, u(0)$ and $u'(0)$ are related by one of the conditions (2.1) or (2.3) where $\xi = u_0$; a and a_0 are some vectors of $\mathbb{C}^k$, and $\alpha \in \mathbb{C}$. In case (2.1) we can rewrite (3.3) in the form

$$\boldsymbol{H}\boldsymbol{u} = \{Au_0 + u(0)m, -u''\}$$

$$A = L + \langle \cdot, a\rangle \tilde{l} : \mathbb{C}^k \to \mathbb{C}^k \qquad m = l + \alpha \qquad \tilde{l} \in \mathbb{C}^k$$

and (3.7) in the form

$$\langle u_0, A^* v_0 - w_0 + v(0)a\rangle + u(0)(\langle m, v_0\rangle + \alpha\overline{v(0)} - \overline{v'(0)}) = 0.$$

Since $u_0 \in \mathbb{C}^k$ and $u(0)$ are arbitrary, this is equivalent to the two relations

$$w_0 = A^* v_0 + v(0)a \qquad v'(0) = \langle v_0, m\rangle + \bar{\alpha} v(0). \tag{3.10}$$

Thus, (3.6) is fulfilled for arbitrary $\boldsymbol{v} \in \mathscr{D}(\boldsymbol{H})$ and $\boldsymbol{w} = \boldsymbol{H}\boldsymbol{v}$ if and only if

$$A = A^* \qquad a = m \qquad \alpha = \bar{\alpha}.$$

Under this assumption $\boldsymbol{H}$ coincides with operator (2.1), (2.2), i.e. $\boldsymbol{H} = H$, and $\boldsymbol{H} \subset \boldsymbol{H}^*$. On the other hand, if (3.6) holds for some $\boldsymbol{v}$ then, according to (3.10), $\boldsymbol{v} \in \mathscr{D}(\boldsymbol{H})$ so that $\boldsymbol{H}$ is self-adjoint.

Similarly, in case (2.3) we rewrite (3.3) in the form

$$\boldsymbol{H}\boldsymbol{u} = \{A_0 u_0 + u'(0)m_0, -u''\}$$

$$A_0 = L - \langle \cdot, a_0\rangle l : \mathbb{C}^k \to \mathbb{C}^k \qquad m_0 = \tilde{l} \in \mathbb{C}^k$$

and (3.7) in the form

$$\langle u_0, A_0^* v_0 - w_0 + v'(0)a_0\rangle + u'(0)(\langle m_0, v_0\rangle + \overline{v(0)}) = 0.$$

Since $u_0 \in \mathbb{C}^k$ and $u'(0)$ are arbitrary, this is equivalent to the two relations

$$\begin{aligned} w_0 &= A_0^* v_0 + v'(0)a_0 \\ v(0) &= -\langle v_0, m_0\rangle. \end{aligned} \tag{3.11}$$

Therefore, (3.6) is fulfilled for arbitrary $\boldsymbol{v} \in \mathscr{D}(\boldsymbol{H})$ if and only if

$$A_0 = A_0^* \qquad a_0 = m_0.$$

Under this assumption $\boldsymbol{H}$ coincides with operator (2.3), (2.4), i.e. $\boldsymbol{H} = H_0$, and $\boldsymbol{H} \subset \boldsymbol{H}^*$. On the other hand, if (3.6) holds for some $\boldsymbol{v}$ then, according to (3.11), $\boldsymbol{v} \in \mathscr{D}(\boldsymbol{H})$ so that $\boldsymbol{H}$ is self-adjoint.

Finally, if $\dim X = k+2$, then $X = X_*$ and the values of u_0, $u(0)$, $u'(0)$ for $\boldsymbol{u} \in \mathscr{D}(\boldsymbol{H})$ are arbitrary. According to (3.3), (3.7) $v(0) = \langle v_0, \tilde{l}\rangle$, $v'(0) = \langle v_0, l\rangle$ so that $v \notin \mathscr{D}(\boldsymbol{H})$ and $\boldsymbol{H}$ is not symmetric. This concludes the proof of theorem 1.

4. Quadratic forms and bound states

Let $y[\cdot,\cdot]$ be a closed positively definite quadratic form with domain $\mathscr{D}[y]$ in some Hilbert space $\mathscr{H}$, $\overline{\mathscr{D}[y]} = H$. By definition, a self-adjoint operator Y corresponds to $y[\cdot,\cdot]$ if $\mathscr{D}(Y) \subset \mathscr{D}[y]$ and $(Yf, g) = y[f, g]$ for every $f \in \mathscr{D}(Y)$ and $g \in \mathscr{D}[y]$. It follows that $\mathscr{D}(Y)$ is dense in $\mathscr{D}[y]$ in the metrics of $y[\cdot,\cdot]$. The lower bounds of y and Y are the same. It can be shown (e.g. see [9]) that there is one-to-one correspondence between positively definite self-adjoint operators and closed quadratic forms. In particular, given an operator Y the form y is constructed as $y[f, g] = (Y^{1/2}f, Y^{1/2}g)$ on domain $\mathscr{D}[y] = \mathscr{D}(Y^{1/2})$. The case of semibounded (from below) operators can always be reduced to the positively definite one by shift by some constant. Quadratic form is a convenient tool for the study of the discrete spectrum. Indeed, for any $\lambda \in \mathbb{R}$ the total multiplicity of the spectrum of an operator Y in the internal $(-\infty, \lambda)$ equals the maximal dimension of such linear sets $\mathcal{M} \subset \mathscr{D}[y]$ that $y[f, f] < \lambda \|f\|^2$ for every $f \in \mathcal{M}, f \neq 0$.

Now, let the operator H be defined by (2.1), (2.2). Integrating by parts we find that for $\boldsymbol{u} = \{\xi, u\} \in \mathscr{D}(H)$

$$h[\boldsymbol{u}, \boldsymbol{u}] = \langle A\xi, \xi\rangle + 2\,\mathrm{Re}\langle \xi, a\rangle \overline{u(0)} + \alpha |u(0)|^2 + \int_0^\infty |u'(r)|^2\, \mathrm{d}r. \tag{4.1}$$

The domain $\mathscr{D}[h]$ is defined as the closure of $\mathscr{D}(H)$ in the metrics

$$h[\boldsymbol{u}, \boldsymbol{u}] + c|\xi|^2 + c\int_0^\infty |u(r)|^2\, \mathrm{d}r \tag{4.2}$$

for sufficiently large $c > 0$. Let $W_2^1 = W_2^1(\mathbb{R}_+)$ be the Sobolev space of functions which belong to L_2 with their first derivatives. Note that for $u \in W_2^1$ the boundary value $u(0)$ is well defined and $|u(0)| \leq C\|u\|_{W_2^1}$ but $u'(0)$ is not, of course, bounded by $\|u\|_{W_2^1}$. Therefore, the metrics, (4.2), is equivalent to $|\xi|^2 + \|u\|^2_{W_2^1}$ and the boundary condition (2.1) disappears by closure of $\mathscr{D}(H)$. It follows that $\mathscr{D}[h] = \mathbb{C}^k \oplus W_2^1$ and representation (4.1) holds for all $\boldsymbol{u} \in \mathscr{D}[h]$.

Similarly, we obtain the expression

$$h_0[\boldsymbol{u}, \boldsymbol{u}] = \langle A_0\xi, \xi\rangle + \int_0^\infty |u'(r)|^2\, \mathrm{d}r \tag{4.3}$$

for the quadratic form of the operator H_0 defined by (2.3), (2.4). The domain $\mathscr{D}[h_0]$ equals the subset of those $\boldsymbol{u} \in \mathbb{C}^k \oplus W_2^1$ for which boundary condition (2.3) holds.

The spectra of the operators H and H_0 consist of the positive continuous parts (this is discussed in section 5) and, possibly, of some numbers of negative eigenvalues. Here we shall show that these numbers equal the numbers of negative eigenvalues of some finite-dimensional matrices. Let us introduce an auxiliary operator $\boldsymbol{A}$ in the space $\mathbb{C}^{k+1}$ by the formula

$$\boldsymbol{A}\boldsymbol{\xi} = \{A\xi + \xi' a, \langle \xi, a\rangle + \alpha\xi'\} \qquad \boldsymbol{\xi} = \{\xi, \xi'\}$$

where $\xi \in \mathbb{C}^k$, $\xi' \in \mathbb{C}$.

Theorem 2. The total number of negative eigenvalues (counted with their multiplicity) of the operator $H(H_0)$ equals the total number of negative eigenvalues of the operator $\boldsymbol{A}(A_0)$ in the space $\mathbb{C}^{k+1}(\mathbb{C}^k)$.

Proof. According to (4.1)

$$h[\boldsymbol{u}, \boldsymbol{u}] = \langle \boldsymbol{A}\boldsymbol{\xi}, \boldsymbol{\xi}\rangle_{\mathbb{C}^{k+1}} + \int_0^\infty |u'(r)|^2\, \mathrm{d}r \qquad \boldsymbol{\xi} = \{\xi, \xi'\} \qquad \xi' = u(0). \tag{4.4}$$

Let us denote by M and $\boldsymbol{m}$ the subspaces in $\mathscr{D}[h]$ and $\mathbb{C}^{k+1}$ spanned by eigenvectors corresponding to negative eigenvalues of the operators H and $\boldsymbol{A}$, respectively. Then $N = \dim M$ and $\boldsymbol{n} = \dim \boldsymbol{m}$ are total numbers of eigenvalues of the operators H and $\boldsymbol{A}$.

First we show that $N \leqslant n$. Let an operator $Q: \mathscr{D}[h] \to \mathbb{C}^{k+1}$ be defined by the relation $Q\{\xi, u(\cdot)\} = \{\xi, u(0)\}$ and let $\hat{M} = QM$ be the image of M under this transformation. According to the relation

$$\langle \mathbf{A}\boldsymbol{\xi}, \boldsymbol{\xi}\rangle_{\mathbb{C}^{k+1}} + \int_0^\infty |u'(r)|^2 \, dr < 0 \qquad \boldsymbol{u} \in M \qquad u \neq 0 \tag{4.5}$$

the equality $Q\boldsymbol{u} = 0$ for $\boldsymbol{u} \in M$ is possible only if $\boldsymbol{u} = 0$. It follows that $\dim \hat{M} = \dim M$. Since, again by (4.5), $\langle \mathbf{A}\boldsymbol{\xi}, \boldsymbol{\xi}\rangle_{\mathbb{C}^{k+1}} < 0$ for every $\boldsymbol{\xi} \in \hat{M}$, $\boldsymbol{\xi} \neq 0$ we have that $n \geqslant \dim \hat{M} = N$.

For the proof of the inequality $N \geqslant n$ we take into account that there exists $\gamma > 0$ such that

$$\langle \mathbf{A}\boldsymbol{\xi}, \boldsymbol{\xi}\rangle_{\mathbb{C}^{k+1}} \leqslant -\gamma |\boldsymbol{\xi}|^2_{\mathbb{C}^{k+1}} \tag{4.6}$$

for every $\boldsymbol{\xi} = \{\xi, \xi'\} \in m$. Denote $\boldsymbol{u}_\varepsilon(r) = \{\xi, \xi'\theta(\varepsilon r)\}$ where θ is some smooth function with compact support and $\theta(0) = 1$. The set M_ε of such $\boldsymbol{u}_\varepsilon$ is linear and dim M_ε = dim m. According to (4.4), (4.6)

$$h[\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon] \leqslant -\gamma(|\xi|^2 + |\xi'|^2) + \varepsilon |\xi'|^2 \int_0^\infty |\theta'(r)|^2 \, dr < 0 \qquad \boldsymbol{\xi} \neq 0$$

for sufficiently small ε. This shows that $N \geqslant n$.

Considering the operator H_0, we shall use a similar notation but add the subscript '0' to distinguish objects related to the operators H_0 and A_0. Let Q_0: $\mathscr{D}[h_0] \to \mathbb{C}^k$ be defined by $Q_0\{\xi, u\} = \xi$ and let $\hat{M}_0 = Q_0 M_0$. It follows from (4.3) that the equality $Q_0\boldsymbol{u} = 0$ for $\boldsymbol{u} \in M_0$ is possible only if $\boldsymbol{u} = 0$ and that $\langle A_0\xi, \xi\rangle < 0$ for all $\xi \in \hat{M}_0$, $\xi \neq 0$. Therefore,

$$n_0 \geqslant \dim \hat{M}_0 = \dim M_0 = N_0.$$

Conversely, for every $\xi \in m_0$ we define $\boldsymbol{u}_\varepsilon(r) = \{\xi, -\langle \xi, a_0\rangle \theta(\varepsilon r)\}$. Such functions satisfy boundary condition (2.3) for any $\varepsilon > 0$ and, by (4.3),

$$h_0[\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon] = \langle A_0\xi, \xi\rangle + \varepsilon |\langle \xi, a_0\rangle|^2 \int_0^\infty |\theta'(r)|^2 \, dr.$$

This quantity is negative for sufficiently small $\varepsilon > 0$ and $\xi \neq 0$ since $\langle A_0\xi, \xi\rangle \leqslant -\gamma|\xi|^2$, $\gamma > 0$, for all $\xi \in m_0$. □

Corollary 1. The total numbers of negative eigenvalues of the operators H and H_0 do not exceed $k+1$ and k, respectively.

The results on the operator H are formulated in terms of the auxiliary operator $\mathbf{A}$. However, some information is available in terms of the operator A only.

Corollary 2. Suppose that the operator A in $\mathbb{C}^k$ has a negative eigenvalue. Then for arbitrary $a \in \mathbb{C}^k$ and $\alpha \in \mathbb{C}$ the operator H has a negative eigenvalue. Moreover, the same conclusion is valid if A has a zero eigenvalue but a is not orthogonal to the corresponding eigenspace of A.

Proof. According to theorem 2 it suffices to show that

$$\langle \mathbf{A}\boldsymbol{\xi}, \boldsymbol{\xi}\rangle_{\mathbb{C}^{k+1}} = \langle A\xi, \xi\rangle + 2\,\mathrm{Re}\langle a, \xi\rangle \xi' + \alpha|\xi'|^2 < 0$$

for some $\boldsymbol{\xi} = \{\xi, \xi'\} \in \mathbb{C}^{k+1}$. Let $A\xi = \lambda\xi$, $\lambda \leqslant 0$, $\xi \neq 0$, and set $\xi' = -\varepsilon\langle \xi, a\rangle$. Then

$$\langle \mathbf{A}\boldsymbol{\xi}, \boldsymbol{\xi}\rangle_{\mathbb{C}^{k+1}} = \lambda|\xi|^2 - \varepsilon(2 - \alpha\varepsilon)|\langle a, \xi\rangle|^2.$$

The right-hand side here is negative for sufficiently small $\varepsilon > 0$ if either $\lambda < 0$ or $\lambda = 0$ but $\langle a, \xi \rangle \neq 0$. □

Below, until the end of this section, we suppose that $k = 1$ so that the vacuum is unique. In this case A is a real and a is a complex constant. By corollary 2, the operator H has a non-trivial negative spectrum for arbitrary $A \leqslant 0$, a and α. Furthermore, according to theorem 2 the number of negative eigenvalues of H is determined by the 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} A & a \\ \bar{a} & \alpha \end{pmatrix} \qquad A \in \mathbb{R} \qquad \alpha \in \mathbb{R} \qquad a \in \mathbb{C}.$$

Finding its eigenvalues we obtain the following result.

Theorem 3. Let $k = 1$. The operator H has two negative eigenvalues if and only if $A + \alpha < 0$ and $|a|^2 < A\alpha$. It has one negative eigenvalue if and only if $|a|^2 > A\alpha$ or $|a|^2 = A\alpha$ and $A + \alpha < 0$. The relation $H \geqslant 0$ is equivalent to the two inequalities $A + \alpha \geqslant 0$ and $|a|^2 \leqslant A\alpha$.

Physically it is reasonable to assume that there is no interaction of the vacuum with itself. This is the case if $A = 0$. For an arbitrary A the problem can be reduced by shift to that without vacuum–vacuum interaction if the operator $-\mathrm{d}^2/\mathrm{d}r^2$ is replaced by $-\mathrm{d}^2/\mathrm{d}r^2 - A$. This procedure makes sense if $A < 0$ so that $-\mathrm{d}^2/\mathrm{d}r^2 - A > 0$. The operator H for $A = 0$, $a = 0$ has the zero eigenvalue at the bottom of the continuous spectrum. Besides, it does not have negative eigenvalues if $\alpha \geqslant 0$. Corollary 2 or theorem 3 show that the zero eigenvalue transforms into an isolated negative eigenvalue as an interaction of a particle with the vacuum is switched on. In the case $A < 0$ the operator H for $a = 0$ has a negative eigenvalue. This eigenvalue cannot be absorbed into the continuous spectrum due to an interaction of a particle with the vacuum.

The results on the operator H_0 are formulated even more simply. Actually, according to theorem 2, H_0 has a negative eigenvalue if and only if $A_0 < 0$. Recall that the Hamiltonian H_0 corresponds to the case where there is no potential zero-range interaction. Thus, if $A_0 \geqslant 0$ an interaction of a particle with the vacuum never binds this particle. On the other hand, the bound state existing for $A_0 < 0$ and $a_0 = 0$ never disappears as an interaction with the vacuum is switched on.

5. The resolvent and the scattering matrix

To construct the resolvent of the operator H we must solve the system of equations

$$A\xi + u(0)a - z\xi = \eta \qquad \mathrm{Im}\, z \neq 0 \qquad \xi, \eta \in \mathbb{C}^k \tag{5.1}$$

$$-u'' - zu = f \qquad u, f \in L_2 \tag{5.2}$$

with boundary condition (2.1). The solution of (5.2) is given by the formula

$$u(r) = \gamma \exp(\mathrm{i}z^{1/2}r) + \mathrm{i}(2z^{1/2})^{-1} \int_0^\infty \exp(\mathrm{i}z^{1/2}|r - r'|) f(r')\, \mathrm{d}r' \qquad \mathrm{Im}\, z^{1/2} > 0 \tag{5.3}$$

so that

$$u(0) = \gamma + \mathrm{i}J \qquad u'(0) = z^{1/2}(\mathrm{i}\gamma + J) \tag{5.4}$$

where

$$J = J(z) = (2z^{1/2})^{-1} \int_0^\infty \exp(iz^{1/2}r) f(r)\, dr. \tag{5.5}$$

Applying the resolvent $R_A(z) = (A - z)^{-1}$ to (5.1) and replacing $u(0)$ by its expression (5.4) we find that

$$\xi = R_A(z)\eta - (\gamma + iJ) R_A(z) a \tag{5.6}$$

and hence

$$\langle \xi, a\rangle + \langle R_A(z)a, a\rangle \gamma = \langle R_A(z)\eta, a\rangle - i\langle R_A(z)a, a\rangle J. \tag{5.7}$$

Substituting (5.4) into boundary condition (2.1) we obtain that

$$\langle \xi, a\rangle + (\alpha - iz^{1/2})\gamma = (z^{1/2} - i\alpha) J. \tag{5.8}$$

Now, subtracting (5.8) from (5.7) we find

$$\gamma = D^{-1}(z)[\langle R_A(z)\eta, a\rangle + (i\alpha - z^{1/2} + i\langle R_A(z)a, a\rangle) J] \tag{5.9}$$

where

$$D(z) = \langle R_A(z)a, a\rangle + iz^{1/2} - \alpha. \tag{5.10}$$

Thus, according to (5.3), (5.6) we have obtained expressions for u and ξ.

Let us rewrite these expressions in matrix notation. It is convenient to introduce the resolvent $\mathscr{R}_0(z)$ of the free operator $-d^2/dr^2$ with the boundary condition $u(0) = 0$ in the space $L_2(\mathbb{R}_+)$. Clearly (cf. (5.3)), $\mathscr{R}_0(z)$ is an integral operator with kernel

$$i(2z^{1/2})^{-1}\{\exp(iz^{1/2}|r - r'|) - \exp[iz^{1/2}(r + r')]\}.$$

Define operators $\Gamma(z) = \Gamma_{A,a}(z) : \mathbb{C}^k \to L_2$ and $\tilde{\Gamma}(z) = \tilde{\Gamma}_{A,a}(z) : L_2 \to \mathbb{C}^k$ by the formulae

$$\begin{aligned} (\Gamma_{A,a}(z)\eta)(r) &= \langle R_A(z)\eta, a\rangle \exp(iz^{1/2}r) \\ \tilde{\Gamma}_{A,a}(z) f &= R_A(z) a \int_0^\infty \exp(iz^{1/2}r) f(r)\, dr \end{aligned} \tag{5.11}$$

and let $K(z)$ be an integral operator with kernel $\exp(iz^{1/2}(r + r'))$. Then, (5.3), (5.6) and (5.9) show that

$$R(z) = \begin{pmatrix} R_A(z) & 0 \\ 0 & \mathscr{R}_0(z) \end{pmatrix} + D^{-1}(z) \begin{pmatrix} -\langle \cdot, R_A(\bar{z})a\rangle R_A(z) a & \tilde{\Gamma}(z) \\ \Gamma(z) & -K(z) \end{pmatrix}. \tag{5.12}$$

Similarly, in the case of operator (2.3), (2.4) the solution of the equation $(H_0 - z)\boldsymbol{u} = \boldsymbol{f}$, $\boldsymbol{u} = \{\xi, u\}$, $\boldsymbol{f} = \{\eta, f\}$, is given by (5.3) and

$$\xi = R_{A_0}(z)\eta - z^{1/2}(i\gamma + J) R_{A_0}(z) a_0$$

where J is defined by (5.5),

$$\gamma = D_0^{-1}(z)[(z^{1/2}\langle R_{A_0}(z)a_0, a_0\rangle - i) J - \langle R_{A_0}(z)\eta, a_0\rangle]$$

and

$$D_0(z) = 1 - iz^{1/2}\langle R_{A_0}(z) a_0, a_0\rangle. \tag{5.13}$$

It follows that the resolvent $R_0(z)=(H_0-z)^{-1}$ has the following representation:

$$R_0(z)=\begin{pmatrix} R_{A_0}(z) & 0 \\ 0 & \mathcal{R}_0(z)\end{pmatrix}$$
$$+D_0^{-1}(z)\begin{pmatrix} \mathrm{i}z^{1/2}\langle\cdot, R_{A_0}(\bar z)a_0\rangle R_{A_0}(z)a_0 & -\tilde\Gamma_0(z) \\ -\Gamma_0(z) & \langle R_{A_0}(z)a_0, a_0\rangle K(z)\end{pmatrix} \qquad (5.14)$$

with $\Gamma_0=\Gamma_{A_0,a_0}$, $\tilde\Gamma_0=\tilde\Gamma_{A_0,a_0}$. Thus, we have obtained the following result.

Theorem 4. The resolvents of the operators H and H_0 are given by (5.12) and (5.14), respectively.

Let us now prove proposition 1. If $A=A_\alpha=A_0+\alpha\langle\cdot, a_0\rangle a_0$, then

$$R_A(z)\eta=R_{A_0}(z)\eta-\alpha\langle R_{A_0}(z)\eta, a_0\rangle(1+\alpha\langle R_{A_0}(z)a_0, a_0\rangle)^{-1}R_{A_0}(z)a_0.$$

In particular, for $a=a_\alpha=\alpha a_0$

$$R_A(z)a=\alpha(1+\alpha\langle R_{A_0}(z)a_0, a_0\rangle)^{-1}R_{A_0}(z)a_0.$$

It follows that

$$R_A(z)a\to R_{A_0}(z)a_0/\langle R_{A_0}(z)a_0, a_0\rangle$$

and

$$D(z)=\mathrm{i}z^{1/2}-\alpha(1+\alpha\langle R_{A_0}(z)a_0, a_0\rangle)^{-1}\to -D_0(z)/\langle R_{A_0}(z)a_0, a_0\rangle$$

as $\alpha\to\infty$. Therefore, comparing (5.12) with (5.14) and taking into account (5.13) we arrive at (2.6).

The resolvents of the operators H and H_0 are analytic functions on the two-sheet Riemannian surface of the function $z^{1/2}$. Poles of $R(z)$ and $R_0(z)$ lying on the first sheet (where $\operatorname{Im} z^{1/2}\geqslant 0$) are real and coincide with eigenvalues of H and H_0, respectively. Poles on the second sheet (where $\operatorname{Im} z^{1/2}<0$) are naturally interpreted as resonances of H or H_0.

Eigenvalues of the operator H are determined by zeros of the function $D(z)$. Indeed, applying (5.12), or by a direct solution of the equation $Hu=zu$, we obtain the following results. Suppose first that $z<0$ is not an eigenvalue of the operator A. Then z is an eigenvalue of the operator H if and only if $D(z)=0$. This eigenvalue is simple and the corresponding eigenvector is

$$u=\gamma\{-R_A(z)a, \exp(\mathrm{i}z^{1/2}r)\} \qquad \gamma\in\mathbb{C}. \qquad (5.15)$$

Suppose now that z is an eigenvalue of A, and denote by $\mathcal{N}_z(A)$ the corresponding eigenspace. Then, for every $\xi\in\mathcal{N}_z(A)$ such that $\langle\xi, a\rangle=0$ the element $u=\{\xi, 0\}$ is an eigenvector of H. In particular, a degenerate eigenvalue of A is also an eigenvalue of H. Furthermore, if the vector a is orthogonal to $\mathcal{N}_z(A)$, $z<0$, then H has an eigenvector of the form (5.15) if and only if $D(z)=0$. Here $R_A(z)$ in (5.10) and (5.15) should be replaced by its regular part at the point z (i.e. the part containing the pole should be omitted). The results for the operator H_0 are formulated quite similarly. In particular, if z does not belong to the spectrum of A_0 then z is an eigenvalue of H_0 if and only if $D_0(z)=0$. The corresponding eigenvector is

$$u=\gamma\{-\mathrm{i}z^{1/2}R_{A_0}(z)a_0, \exp(\mathrm{i}z^{1/2}r)\}.$$

In the case $k=1$ the equation $D_0(z)=0$ is quadratic with respect to $s=-\mathrm{i}z^{1/2}$ and can be solved explicitly. Indeed, it has the roots

$$2s_\pm = -|a_0| \pm (|a_0|^2 - 4A_0)^{1/2}.$$

Roots s lying on the positive half-axis correspond to eigenvalues $z=-s^2$ of H and roots s in the left half-plane correspond to its resonances $z=-s^2$. Let us analyse the dependence of $s_\pm$ on the coupling constant a_0 of the interaction of a particle with the vacuum. Note that for $a_0=0$ the operator $H_0 = H_0(A_0, 0)$ has the eigenvalue A_0 (the corresponding function $D_0(z)$ has poles on both sheets of the Riemannian surface of the function $z^{1/2}$). If $A_0=0$, the zero eigenvalue of $H_0(A_0, 0)$ is split up for $a_0 \neq 0$ into the negative eigenvalue $-|a_0|^2$ and the zero-energy resonance at the bottom of the continuous spectrum. If $A_0<0$, the negative eigenvalue A_0 transforms for $a_0\neq 0$ into the negative eigenvalue $\lambda = -s_+^2$ and the resonance at the point $-s_-^2$. Finally, in the case $A_0>0$, the positive eigenvalue A_0 creates two resonances $-s_\pm^2$ which are complex and mutually conjugated if $|a_0|^2 < 4A_0$ and are negative if $|a_0|^2 \geq 4A_0$.

The equation $D(z)=0$ is cubic (for $k=1$) with respect to $z^{1/2}$ and also allows one to study the behaviour of eigenvalues and resonances of the operator H with respect to parameters A, a and α. In terms of $s=-\mathrm{i}z^{1/2}$ this equation reads

$$(s^2+A)(s+\alpha) - |a|^2 = 0. \tag{5.16}$$

We describe only the dependence of eigenvalues of $H=H(A, a, \alpha)$ as $a\to 0$ for fixed A and α. Let first $A=0$. Then, by (5.16) the operator H has exactly one negative eigenvalue λ for small a (this is also a consequence of theorem 3) and $\lambda \sim -|a|^{4/3}$ for $\alpha=0$, $\lambda\sim -\alpha^{-1}|a|^2$ for $\alpha>0$ and $\lambda \sim -\alpha^2 + 2\alpha^{-1}|a|^2$ for $\alpha<0$. If $A>0$, then H has a negative eigenvalue λ only for $\alpha<0$. In this case

$$\lambda = -\alpha^2 + 2\alpha(A+\alpha^2)^{-1}|a|^2. \tag{5.17}$$

Finally, if $A<0$, then H has only one eigenvalue

$$\lambda \sim A - (|A|^{1/2} + \alpha)^{-1}|a|^2 \tag{5.18}$$

for $\alpha \geq 0$ and two eigenvalues λ_1, λ_2 for $\alpha<0$. If $|A| \neq \alpha^2$, then one of them has the asymptotics (5.17), and another the asymptotics (5.18). If $|A|=\alpha^2$, then

$$\lambda_{1,2} \sim A - (2|\alpha|)^{1/2}|a|.$$

The explicit representations for the resolvents allow one to construct the spectral family of the operators H and H_0, the expansions in eigenfunctions and so forth. Since, however, this is quite similar to the same procedures for the standard zero-range interaction (without the interaction with the vacuum), we will not dwell upon it.

Let us calculate only the scattering matrix for the Hamiltonians H and H_0. Note previously that all basic objects of the scattering theory are well defined in our case. We can choose for an unperturbed operator $H^{(0)}$ the operator $H_0(A_0, 0)$ for any A_0. Since the absolutely continuous part of $H_0(A_0, 0)$ does not depend on A_0 any choice is possible. For instance, we can set $A_0=0$. The wave operators for the pair $H^{(0)}$, H exist, are complete and the scattering matrix $S(\lambda)$: $\mathbb{C}\to\mathbb{C}$, $\lambda>0$ is defined by the following standard procedure. One looks for the solution of the equations

$$A\xi + u(0)a = \lambda\xi \qquad -u'' = \lambda u$$

satisfying boundary condition (2.1) and having the form

$$u(r) = \exp(-\mathrm{i}\lambda^{1/2}r) - S(\lambda)\exp(\mathrm{i}\lambda^{1/2}r).$$

Similarly to the solution of (5.1), (5.2) we find that

$$S(\lambda)=D(\lambda-\mathrm{i}0)D^{-1}(\lambda+\mathrm{i}0) \qquad D(\lambda-\mathrm{i}0)=\overline{D(\lambda+\mathrm{i}0)} \tag{5.19}$$

if $\lambda\notin\sigma(A)$ or $\lambda\in\sigma(A)$ but a is orthogonal to $\mathcal{N}_\lambda(A)$. Just in the same way we find that the scattering matrix $S_0(\lambda)$ for the pair $H^{(0)}$, H_0 is

$$S_0(\lambda)=D_0(\lambda-\mathrm{i}0)D_0^{-1}(\lambda+\mathrm{i}0) \qquad D_0(\lambda-\mathrm{i}0)=\overline{D_0(\lambda+\mathrm{i}0)}, \tag{5.20}$$

if $\lambda\notin\sigma(A)$ or $\lambda\in\sigma(A_0)$ but a_0 is orthogonal to $\mathcal{N}_\lambda(A_0)$. If $\lambda\in\sigma(A)(\lambda\in\sigma(A_0))$ and $a(a_0)$ has a non-trivial projection onto $\mathcal{N}_\lambda(A)(\mathcal{N}_\lambda(A_0))$, then $S(\mu)\to 1(S_0(\mu)\to 1)$ as $\mu\to\lambda$ (from both sides). Thus, the functions $S(\lambda)$ and $S_0(\lambda)$ are continuous in $\lambda>0$. Furthermore, $S(\lambda)\to -1$ and $S_0(\lambda)\to 1$ as $\lambda\to\infty$. Recall that for a zero-range potential well (without an interaction with the vacuum) the scattering matrix tends to -1 as $\lambda\to\infty$. The fact that $S_0(\lambda)\to 1$ as $\lambda\to\infty$ is not surprising because for the Hamiltonian H_0 the zero-range potential well equals zero.

6. The three-dimensional representation

As was noted in the introduction, the study of an interaction of a three-dimensional particle with a zero-range potential well and the vacuum can be reduced to that for a particle on the half-axis. Actually, let $\mathscr{H}=\mathbb{C}^k\oplus L_2(\mathbb{R}^3)$ now, so that elements of $\mathscr{H}$ are couples $\boldsymbol{u}=\{\xi,u\}$, $\xi\in\mathbb{C}^k$, $u\in L_2(\mathbb{R}^3)$. Hamiltonians are introduced as self-adjoint extensions of the symmetric operator $\mathring{H}=-\Delta$ defined on the domain of elements $\boldsymbol{u}=\{0,u\}$, $u\in C_0^\infty(\mathbb{R}^3\backslash\{0\})$. Let us introduce spherical coordinates $r=|x|$, $\omega=x|x|^{-1}$ in $\mathbb{R}^3$ and set

$$u(x)=\sum_{l=0}^{\infty} r^{-1}u_l(r)Y_l(\omega)$$

where Y_l is the spherical function. Since

$$(\mathring{H}u)(x)=\sum_{l=0}^{\infty} r^{-1}[-u_l''(r)+l(l+1)r^{-2}u_l(r)]Y_l(\omega)$$

the operator $\mathring{H}$ in the space $L_2(\mathbb{R}^3)$ is unitarily equivalent to the orthogonal sum of the operators

$$\mathring{H}_l=-\mathrm{d}^2/\mathrm{d}r^2+l(l+1)r^{-2} \qquad l=0,1,2,\ldots$$

acting in the space $L_2(\mathbb{R}_+)$ and defined on $C_0^\infty(\mathbb{R}_+)$. The operators $\mathring{H}_l$ for $l\geqslant 1$ are essentially self-adjoint. Therefore, the problem is reduced to construction of self-adjoint extensions of the operator $\mathring{H}_0$ in the space $\mathbb{C}^k\oplus L_2(\mathbb{R}_+)$.

This problem was considered in theorem 1. Now we need only to rewrite its results in terms of the space $\mathbb{C}^k\oplus L_2(\mathbb{R}^3)$. In this representation the role of the set $\mathbb{D}_*=\mathbb{C}^k\oplus W_2^2(\mathbb{R}_+)$ is played by

$$\mathscr{D}_*=\mathbb{C}^k\oplus \tilde{W}_2^2(\mathbb{R}^3)$$

where $\tilde{W}_2^2(\mathbb{R}^3)$ is defined as follows. A function $u\in\tilde{W}_2^2(\mathbb{R}^3)$ if u belongs to the space $W_2^2(\mathbb{R}^3)$ outside any neighbourhood of the point $x=0$ and it admits as $x\to 0$ the representation

$$u(x)=u^{(0)}|x|^{-1}+u^{(1)}+v(x)$$

where $v \in W_2^2(\mathbb{R}^3)$ and $v(0)=0$. Let the operator $-\Delta$ be defined on $\tilde{W}_2^2(\mathbb{R}^3)$ in the sense of distributions with the set of test functions being $C_0^\infty(\mathbb{R}^3\backslash\{0\})$. The following assertion is a direct reformulation of theorem 1.

Theorem 5. Suppose that A, A_0, a, a_0 and α are the same as in theorem 1. Let $\mathscr{D}(H)\subset\mathscr{D}_*$ consist of elements $\boldsymbol{u}=\{\xi, u\}$ such that

$$u^{(1)}=\langle\xi, a\rangle+\alpha u^{(0)}.$$

Then the operator H defined by the formula

$$H\boldsymbol{u}=\{A\xi+u^{(0)}a, -\Delta u\}$$

is self-adjoint and $\mathring{H}\subset H$. Let $\mathscr{D}(H_0)\subset\mathscr{D}_*$ consist of elements $\boldsymbol{u}$ such that

$$u^{(0)}=-\langle\xi, a_0\rangle.$$

Then the operator H_0 defined by the formula

$$H_0\boldsymbol{u}=\{A_0\xi+u^{(1)}a_0, -\Delta u\}$$

is self-adjoint and $\mathring{H}\subset H_0$. On the other hand, any self-adjoint extension of the operator $\mathring{H}$ in the space $\mathscr{H}$ has one of these two forms.

All remarks made in section 2 are relevant for operators H_0 and H considered in the space $\mathbb{C}^k\oplus L_2(\mathbb{R}^3)$. In particular, proposition 1 also holds in this representation.

The resolvents $R(z)$ and $R_0(z)$ of the operators H and H_0 are again given by (5.12) and (5.14). Now $\mathscr{R}_0(z)$ and $K(z)$ are the integral operators with kernels

$$(4\pi)^{-1}|x-x'|^{-1}\exp(\mathrm{i}z^{1/2}|x-x'|)$$

and

$$(4\pi)^{-1}(|x||x'|)^{-1}\exp(\mathrm{i}z^{1/2}(|x|+|x'|)$$

respectively. The functions $D(z)$ and $D_0(z)$ are, as before, defined by (5.10) and (5.13) and $\Gamma=\Gamma_{A,a}$, $\tilde{\Gamma}=\tilde{\Gamma}_{A,a}$, $\Gamma_0=\Gamma_{A_0,a_0}$, $\tilde{\Gamma}_0=\tilde{\Gamma}_{A_0,a_0}$. According to (5.11), the operators $\Gamma_{A,a}(z)$: $\mathbb{C}^k\to L_2(\mathbb{R}^3)$ and $\tilde{\Gamma}_{A,a}(z): L_2(\mathbb{R}^3)\to\mathbb{C}^k$ are determined by the equalities

$$(\Gamma_{A,a}(z)\eta)(x)=2^{-1}\pi^{-1/2}\langle R_A(z)\eta, a\rangle|x|^{-1}\exp(\mathrm{i}z^{1/2}|x|)$$

$$\tilde{\Gamma}_{A,a}(z)f=2^{-1}\pi^{-1/2}R_A(z)a\int_{\mathbb{R}^3}|x|^{-1}\exp(\mathrm{i}z^{1/2}|x|)f(x)\,\mathrm{d}x.$$

Finally, we note that (5.19), (5.20) give the following expressions for the scattering amplitudes. They do not depend on angular variables and are

$$F(\lambda)=-(\mathrm{i}\lambda^{1/2}-\alpha+\langle R_A(\lambda)a, a\rangle)^{-1} \qquad \lambda\notin\sigma(A),$$

for the Hamiltonian H and

$$F_0(\lambda)=\langle R_{A_0}(\lambda)a_0, a_0\rangle(1-\mathrm{i}\lambda^{1/2}\langle R_{A_0}(\lambda)a_0, a_0\rangle)^{-1} \qquad \lambda\notin\sigma(A_0)$$

for the Hamiltonian H_0.

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